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GRADIENTS OF LOCAL LINEAR HULLS IN FINITE-DIFFERENCE OPERATORS FOR THE HAMILTON–JACOBI EQUATIONS†

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A finite-difference operator (FDO) for the Hamilton–Jacobi equation is presented in which the non-existent solution gradients are replaced by the gradients of linear hulls. The approximation scheme (AS) corresponding to this FDO is proved to be minorized and majorized by ASs with FDOs based on the construction of sub-differentials and superdifferentials of local convex and concave hulls. This makes it possible to verify that the ASs converge to the linear constructions. Modifications of the FDO taking into account the configuration of local attainability domains are considered. The results of numerical experiments are presented.

1. FORMULATION OF THE PROBLEM

We consider the problem of constructing a finite-difference operator (FDO) based on linear constructions for the approximation schemes (AS) for solving the Hamilton–Jacobi equation. We use the results of the theory of optimal guaranteed control [1, 2], convex and non-smooth analysis [3, 4] and the general theory of the generalized (minimax, viscosity) solution of first-order partial differential equations of Hamilton–Jacobi type [5, 6].

FDOs of various types have been developed in [7–12]. In particular [11, 12], FDOs with subdifferentials and superdifferentials of local convex and concave hulls have been considered and the convergence of the corresponding ASs has been proved. Such ASs have a number of merits. Besides, the construction of local convex and concave hulls is a laborious computational procedure requiring complex programs and considerable computer resources. It is therefore topical to apply linear constructions to FDOs and to verify the convergence of the corresponding ASs.

We propose to use local linear hulls of solutions constructed by the method of least squares. The proof of the convergence of the AS with a FDO of this type is based on verifying order relationships linking local linear hulls with local convex (concave) hulls. Note the simplicity of the realizations of the formulae of the method of least squares when constructing the gradients of local linear hulls as compared with the algorithms for computing the subdifferentials and superdifferentials of local convex and concave hulls [11, 12].

We will consider modifications of the FDO with subdifferentials and superdifferentials of convex and concave hulls and with gradients of local hulls in which the local constructions are carried out in neighbourhoods as close as possible to the attainability domains. Such FDOs reduce computing costs considerably compared with FDOs in neighbourhoods symmetric about the centre of the attainability domain.

The proposed AS has been used to solve a number of problems of guaranteed control using the formulations in [13–16]. The solution of an evolutionary game [15, 16] will be presented, which demonstrates the efficiency of the AS with gradients of local linear hulls.

2. THE PROBLEM OF GUARANTEED CONTROL AND THE BOUNDARY-VALUE PROBLEM FOR THE HAMILTON–JACOBI EQUATION

We consider the Cauchy problem for the Hamilton–Jacobi equation

$$\frac{\partial w}{\partial t}+H\left(t,\ x,\ \frac{\partial w}{\partial x}\right)=0,\ (t,\ x)=(t_0,\ \theta)\times R^n \tag{2.1}$$

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$$w(\theta,\ x)=\sigma(x),\ x\in R^n \tag{2.2}$$

We will assume that, corresponding to this boundary-value problem, we have the problem of guaranteed control for a dynamic system

$$\dot{x}=f(t,\ x,\ u,\ v)=h(t,\ x)+b(t,\ x)u+c(t,x)v \tag{2.3}$$

$$t\in T=[t_0,\ \theta],\ x\in R^n,\ u\in P\subset R^p,\ v\in Q\subset R^q$$

with terminal payoff functional

$$\gamma(x(\cdot))=\sigma(x(\theta)) \tag{2.4}$$

Here x is the n-dimensional vector of the system, u is the control interaction and v is the noise vector. The sets P and Q are convex and compact.

The function $H(t,x,s):T\times R^n\times R^n\to R$ in (2.1) is the Hamiltonian of system (2.3), i.e. it is related to the dynamics $f(t,x,u,v)$ by the relation

$$H(t,\ x,\ s)=\langle s,\ h(t,\ x)\rangle+\min_{u\in P}\langle s,\ b(t,\ x)u\rangle+\max_{v\in Q}\langle s,\ c(t,x)v\rangle \tag{2.5}$$

Suppose that the right-hand side $f(t,x,u,v)$ of (2.3) satisfies the conventional conditions: Lipschitz continuity in t,x and the extendability condition for solutions.

A compact domain $G_r\in T\times R^n$, in which to consider (2.1) and (2.3), will be defined by the invariance condition: if $(t_*,x_*)\in G_r$, then $(t,x_*+(t-t_*)B_r)\in G_r$ for all $t\in[t_*,\theta]$, $B_r=\{b\in R^n=\{b\in R^n:\ ||\,b\,||\leq r\}$, where $r>K$, and

$$K=\max_{(t,\ x,\ u,\ v)\in G\times P\times Q}\|f(t,\ x,\ u,\ v)\| \tag{2.6}$$

is the velocity of the system defined on a closed set G that satisfies the strong invariance condition under the differential inclusion

$$\dot{x}(t)\in F(t,\ x(t)),\ t\in[t_*,\ \theta],\ x(t_*)=x_*$$

$$F(\tau,\ y)=\{f\in R^n:\ f=f(\tau,\ y,\ u,\ v),\ u\in P,\ v\in Q\},\ (\tau,\ y)\in T\times R^n$$

It is clear that $G_r\subset G$.

By the above conditions for the right-hand side of (2.3) the Hamiltonian $H(t,x,s):G_r\times R^n\to R$ is Lipschitz continuous and positively homogeneous in s.

A fundamental role in the solution of (2.3), (2.4) is played by the value function $(t,x)\to w(t,x)$: $G_r\to R$, defined for the initial position (t_*,x_*), the positional strategies $U=U(t,x)$ and the corresponding trajectories $x(\cdot)\in X(t_*,x_*,U)$ by the formula

$$w(t_*,\ x_*)=\min_U\max_{x(\cdot)\in X(t_*,\ x_*,\ U)}\sigma(x(\theta)) \tag{2.7}$$

Optimal strategies can be constructed, for example, by the method of extrapolation shift to the corresponding points of the local extrema of w [1]. We know [5] that the value function w is a generalized maximal solution of the boundary-value problem (2.1), (2.2). Therefore, to construct the value function one can use an AS for the Hamilton–Jacobi equation.

3. CONVEX, CONCAVE AND LINEAR HULLS OF PIECEWISE-CONTINUOUS FUNCTION

Here we will solve an auxiliary problem concerned with relations linking local convex (concave) and linear hulls.

In the metric

$$\rho_1(x,\ y)=\max_{i\in 1..n}\left|x^i-y^i\right| \tag{3.1}$$

we consider the n-dimensional neighbourhood $\overline{O}_{\rho 1}(x, r_1) = \{y, \rho_1(x, y) \leq r_1, x, y \in R^n\}$.

The neighbourhood $\overline{O}_{\rho 1}(x, \tau_1)$ is an n-dimensional cube on which we take a uniform mesh with step $\delta = r_1/N$. The number of points of the division along each axis is equal to $2N + 1$. The total number of mesh nodes will be denoted by M, $M = (2N + 1)^n$.

We observe that in this case the radius of the neighbourhood is related to the division step by

$$r_1 = N\delta \tag{3.2}$$

Let a tabulated function

$$\bar{U} = \{(y_k, u(y_k)) : \quad k = 1, \dots, M\} \tag{3.3}$$

be given at the nodes y_k of the uniform mesh. For this function we define the following constructions.

$L(y)$: $\overline{O}_\rho(x, r_1) \to R$—the hyperplane closest to $\bar{U}$ in the sense of square deviation,

$f(y)$: $\overline{O}_\rho(x, r_1) \to R$—the convex hull for $\bar{U}$,

$g(y)$: $\overline{O}_\rho(x, r_1) \to R$—the concave hull for $\bar{U}$.

We observe that in a neighbourhood of smaller radius, which will be denoted by $\overline{O}_\rho(x, r_2) \to R$, the hyperplane $L(y)$ lies between the convex and concave hulls. The following assertion provides the precise ratio of the radii of the neighbourhoods $\overline{O}_\rho(x, r_1)$ and $\overline{O}_\rho(x, r_2)$ under consideration.

Theorem 3.1. Let the radii r_1 and r_2 of $\overline{O}_\rho(x, r_1)$ and $\overline{O}_\rho(x, r_2)$ be chosen in such a way that

$$\frac{r_1}{r_2} = 3n\left(1 - \frac{1}{N+1}\right) \tag{3.4}$$

Then

$$f(y) \leq L(y) \leq g(y), \quad y \in \overline{O}_{\rho_1}(x, r_2) \tag{3.5}$$

Proof. The local convex (concave) hulls of $u(y)$ in the closed neighbourhood $\overline{O}_\rho(x, r_1)$ of x of radius r can be defined [3] as the lower and upper bounds

$$f(y) = \inf\{\sum_k \beta_k u(y_k) : \quad \sum_k \beta_k y_k = y, \ \sum_k \beta_k = 1, \ \beta_k \geq 0\} \tag{3.6}$$

$$y_k \in \overline{O}_{\rho_1}(x, r_1), \quad y \in \overline{O}_{\rho_1}(x, r_1)$$

$$g(y) = \sup\{\sum_k \gamma_k u(y_k) : \quad \sum_k \gamma_k y_k = y, \ \sum_k \gamma_k = 1, \ \gamma_k \geq 0\} \tag{3.7}$$

$$y_k \in \overline{O}_{\rho_1}(x, r_1), \quad y \in \overline{O}_{\rho_1}(x, r_1)$$

Here and everywhere henceforth $k = 1, \dots, M$.

We also consider the linear hull

$$L(y) = \langle A, y \rangle + B \tag{3.8}$$

of $\bar{U}$ in $\overline{O}_\rho(x, r_1)$. The parameters $A \in R^n$, $B \in R$ can be determined from the condition for a minimum of the quadratic deviation of $L(y)$ from the tabulated values of $\bar{U}$

$$\min_{A,\, B} \sum_k [u(y_k) - (\langle A, y_k \rangle + B)]^2$$

This condition leads to a system of linear equations for A and B

$$\sum_k \langle A, y_k - x \rangle (y_k - x) = \sum_k u(y_k)(y_k - x) \tag{3.9}$$

$$B = -\langle A, x \rangle + \frac{1}{M} \sum_k u(y_k) \tag{3.10}$$

$L(Y)$ can be represented by

$$L(y) = \frac{1}{M}\sum_k u(y_k) + \frac{1}{\delta^2 a}\sum_{i=1}^{n}(y^i - x^i)\sum_k (y_k^i - x^i)u(y_k)$$

$$a = (2N+1)^n (N+1)N/3$$

Using the fact that the mesh is uniform in $\overline{O}_{\rho 1}(x, r_1)$, we transform $L(y)$ to the form

$$L(y) = \frac{1}{M}\sum_k u(y_k) + \sum_{i=1}^{n}\left[\frac{3\delta(y^i - x^i)}{M(N+1)N}\sum_{j=-N} j\sum_{s=1}^{S} u(y_s)\right] \tag{3.11}$$

$$y_s = y_s(i,j) = (y_s^1, \ldots, x^i + j\delta, \ldots, y_s^n)$$

$$y_s^l = x^i + m\delta, \quad m = -N, \ldots, N, \quad l \neq i, \quad S = (2N+1)^{n-1}$$

We observe that the structure of (3.11) is similar to (3.6) and (3.7) and can be represented as a linear combination

$$L(y) = \sum_k \alpha_k u(y_k), \quad y_k \in \overline{O}_{\rho_1}(x, r_1) \tag{3.12}$$

$$\alpha_k = \alpha_k(y) = \frac{1}{M} + \sum_{i=1}^{n}\frac{y^i - x^i}{\delta a} j_k^i, \quad j_k^i = \frac{y_k^i - x^i}{\delta}$$

It can be shown that the coefficients α_k in (3.12) satisfy the following two conditions from definition (3.6) and (3.7) of convex and concave hulls

$$\sum_k \alpha_k y_k = y, \quad \sum_k \alpha_k = 1, \quad y \in \overline{O}_{\rho_1}(x, r_1)$$

However, the third condition in (3.6) and (3.7) fails to be satisfied everywhere in the neighbourhood $\overline{O}_\rho(x, r_1)$, i.e. the coefficients α_k may fail to be non-negative there.

We shall specify a smaller neighbourhood $\overline{O}_\rho(x, r_2)$ in which all the functions $\alpha_k(y)$ in (3.12) will be non-negative. Using the symmetry of $\overline{O}_\rho(x, r_1)$ about the centre, we transform the system of inequalities $\alpha_k \geq 0$ to the form

$$\left|\sum_{i=1}^{n} j_l^i (y^i - x^i)\right| < (N+1)N\delta/3, \quad l = 1..M/2$$

The resulting system can be reduced to the equivalent inequality

$$\sum_{i=1}^{n}\left|y^i - x^i\right| \leq \frac{(N+1)\delta}{3} = r_3$$

which defines a neighbourhood $\overline{O}_{\rho 2}(x, r_3)$ of radius r_3 and centre x in the metric ρ_2, where

$$\rho_2(x, y) = \sum_{i=1}^{n}\left|x^i - y^i\right| \tag{3.13}$$

If we return to the original metric ρ_1, the radius of the neighbourhood will be reduced by a factor of n

$$r_2 = r_3/n \tag{3.14}$$

Thus, in $\overline{O}_{\rho 1}(x, r_2)$ the coefficients α_k satisfy all three conditions in (3.6) and (3.7). From (3.6), (3.7) and (3.12) it follows that the linear combination $L(y)$ lies between the lower bound $f(y)$ and the upper bound $g(y)$. Using (3.2) and (3.14), we find that the radii of $\overline{O}_\rho(x, r_1)$ and $\overline{O}_{\rho 1}(x, r_2)$ are related by

$$\frac{r_1}{r_2}=3n\left(1-\frac{1}{N+1}\right)$$

Remark 3.1. In computations it is more efficient to construct $L(y)$ in the neighbourhood

$$\bar{O}_{\rho_2}(x,\ r_1)=\{y,\ \rho_2(x,\ y)\le r_1,\ x,\ y\in R^n\}$$

defined in the dual norm ρ_2. In this case the constructions involve nodes from a much smaller set (for example, for $n = 2$ the number of nodes is reduced by a factor of 2). Relation (3.4) between the radii r_1 and r_2 of $\bar{O}_\rho(x, r_1)$ and $\bar{O}_\rho(x, r_2)$ is preserved.

4. LINEAR FINITE-DIFFERENCE OPERATOR FOR THE HAMILTON–JACOBI EQUATION

We will now solve the above problem of constructing the value function $w(t, x)$, which is a solution of boundary-value problem (2.3), (2.4). We shall define the FDO for the Hamilton–Jacobi equation. Let the discretization step Δ of the interval T be given along with times $t, t + \Delta \in T$. Suppose that at $t + \Delta$ a Lipschitz-continuous function $x \to u(t + \Delta, x) = u(x)$ (with Lipschitz constant L) is defined, which approximates the generalized solution $x \to w(t + \Delta, x)$. The values of the function $x \to u(t, x) = v(x)$ approximating the solution $x \to w(t, x)$ at time t can be defined as the values of the FDO LA as follows:

$$v(x)=LA(t,\ \Delta,\ u)(x)=u_0+\Delta H(t,\ x,\ A),\ u_0=\frac{1}{M}\sum_k u(y_k) \tag{4.1}$$

where $u(y_k)$ are the values of the tabulated function $\bar{U}$ in (3.3) at the nodes $y_k \in \bar{O}_{\rho 1}(x, r\Delta)$ and A is the gradient of the linear function $L(y)$ defined by system (3.9), (3.10) for $r_1 = r\Delta$.

Theorem 4.1. Let r and K be related by the relation

$$\frac{r}{K}=3n\left(1-\frac{1}{N+1}\right) \tag{4.2}$$

Then the AS with the FDO $u \to LA(t, \Delta, u)$ converges with convergence rate $\Delta^{1/2}$.

Proof. The proof of convergence will be based on the order relationship between the hypersurface and the convex and concave hulls (3.5) for $r_1 = K\Delta$, $r_2 = r\Delta$. To this end we state the formulae for the FDOs of the local convex (concave) hulls

$$\begin{aligned} v_*(x)=F_1(t,\ \Delta,\ u)(x)=f(x)+&\max_{y\in\bar{O}(x,\ K\Delta)}\ \max_{s\in D_*f(y)}(\Delta H(t,\ x,\ s)+\\ &+f(y)-f(x)-\langle s,\ y-x\rangle)\end{aligned} \tag{4.3}$$

$$\begin{aligned} v^*(x)=F_2(t,\ \Delta,\ u)(x)=g(x)+&\min_{y\in\bar{O}(x,\ K\Delta)}\ \min_{s\in D^*g(y)}\{(\Delta H(t,\ x,\ s)+\\ &+g(y)-g(x)-\langle s,\ y-x\rangle)\}\end{aligned} \tag{4.4}$$

Here $f(y)$: $\bar{O}(x, r\Delta) \to R$ is the local convex hull of $u(y) = u(t + \Delta, y)$ in the closed neighbourhood $\bar{O}(x, r\Delta)$ of x of radius $r\Delta$, $r > K$. The set $D_*f(y)$ is the subdifferential of $f(y)$ at the point y

$$D_*f(y)=\{s\in R^n: f(z)-f(y)\ge\langle s,\ z-y\rangle,\ z\in\bar{O}(x,\ r\Delta)\},\ y\in\bar{O}(x,\ K\Delta)$$

The function $g(y)$: $\bar{O}(x, r\Delta) \to R$ is the locally convex hull of $u(y)$. The set $D^*g(y)$ is the superdifferential of $g(y)$ at the point y

$$D^*g(y)=\{s\in R^n: g(z)-g(y)\le\langle s,\ z-y\rangle,\ z\in\bar{O}(x,\ r\Delta)\},\ y\in\bar{O}(x,\ K\Delta)$$

Note that the relations

$$LA(t,\ \Delta,\ u)(x) = F_1(t,\ \Delta,\ L)(x) = F_2(t,\ \Delta,\ L)(x) \tag{4.5}$$

are satisfied for the aforementioned operators. It was proved in [11, 12] that F_1 and F_2 are monotone

$$F_i(t,\ \Delta,\ u_1)(y) \le F_i(t,\ \Delta,\ u_2)(y) \tag{4.6}$$

if $u_1(y) \leqslant u_2(y) \leqslant F_i(t, \Delta, u_2)(y)$. It follows from (3.4), (4.5) and (4.6) that

$$F_1(t,\ \Delta,\ u)(x) \le LA(t,\ \Delta,\ u)(x) \le F_2(t,\ \Delta,\ u)(x) \tag{4.7}$$

It has also been proved [11, 12] that the ASs with the operators F_1 and F_2 converge with convergence rate $\Delta^{1/2}$. Then (4.7) implies the convergence of the AS with the FDO (4.1) based on the gradients of local linear hulls.

Remark 4.1. The FDO obtained above can be interpreted on the elementary n-dimensional rhombus $R(x, \delta) = \{y \in R^n\colon \rho_2(x, y) < \delta\}$ in phase space in the metric ρ_2 defined by (3.13) as the Lax–Friedrichs operator [7]

$$LF(t,\ \Delta,\ u) = w(x) + \Delta H(t,\ x,\ a)$$

$$a = (a_1, \ldots,\ a_n),\ a_i = \frac{u(x+\delta e_i) - u(x-\delta e_i)}{2\delta},\ nK\Delta < \delta$$

e_i $(i = 1, \ldots, n)$ are the unit vectors.

5. OPERATORS WITH SUBDIFFERENTIALS AND SUPERDIFFERENTIALS OF LOCAL CONVEX AND CONCAVE HULLS WITH GRADIENTS OF LOCAL LINEAR HULLS IN ASYMMETRIC NEIGHBOURHOODS

Note that the FDOs LA, F_1 and F_2 in (4.1), (4.3) and (4.4) are given in symmetric neighbourhoods whose dimensions are determined by the maximum speed of system (2.6) and can be as large as desired. The FDOs do not take into account the position in a symmetric neighbourhood of the attainability domain $D(t, x, \Delta) = \{y \in R^n\colon y = x + \Delta f(t, x, u, v), u \in P, v \in Q\}$.

Remark 5.1. The FDOs F_1 and F_2 in (4.3) and (4.4) are independent of the symmetry of the neighbourhood and can be used for computations in any neighbourhood containing the attainability domain $D(t, x, \Delta)$. In the attainability domain itself the operators have the form

$$FD_1(t,\ \Delta,\ u)(x) = \max_{y \in D(t,\ x,\ \Delta)} \max_{s \in D_* f(y)} (\Delta H(t,\ x,\ s) + f(y) - \langle s,\ y - x\rangle)$$

$$FD_2(t,\ \Delta,\ u)(x) = \min_{y \in D(t,\ x,\ \Delta)} \min_{s \in D^* g(y)} (\Delta H(t,\ x,\ s) + g(y) - \langle s,\ y - x\rangle)$$

Here $y \to f(y)\colon \bar{O}(x, r_1\delta) \to R$ is the convex hull of $y \to u(y)$ in $\bar{O}(x, r_1\delta)$, $D(t, x, \delta) \subset \bar{O}(x, r_1\delta)$ and $y \to g(y)$: $\bar{O}(x, r_1\delta) \to R$ is the concave hull of this function.

The FDO LA in (4.1) can also be extended to the case of the closest neighbourhood to the attainability domain $D(t, x, \Delta)$ of system (2.3), (2.4), which can considerably reduce the amount of computation. We consider an n-dimensional rectangle containing the attainability domain $D(t, x, \Delta)$

$$P(t,\ x,\ \Delta) = \{y \in R^n\colon\ s^i \le y^i \le S^i,\ i = 1, \ldots, n$$

$$s^i < \min_{d \in D(t,\ x,\ \Delta)} d^i,\ S^i > \max_{d \in D(t,\ x,\ \Delta)} d^i\}$$

Equivalently, the neighbourhood $P(t, x, \Delta)$ can be defined by

$$P(t,\ x,\ \Delta) = P(\bar{x},\ q) = \{y \in R^n:\ \left|\bar{x}^i - y^i\right| \le q^i,\ i = 1,\dots,n\}$$

Here $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ is the centre and $q = (q^1, \dots, q^n)$ are the dimensions of the neighbourhood $P(\bar{x}, r)$.

Assumption 5.1. Let $L(y) = \langle A, y\rangle + B$ be the local linear hull constructed by the method of least squares in $P(\bar{x}, r)$, where $r = r^1, \dots, r^n$ is given by

$$\frac{r^i}{q^i} = 3n\left(1 - \frac{1}{N^i + 1}\right),\ N^i = \frac{r^i}{\delta} \tag{5.1}$$

Then the FDO LA in (4.1) can be modified in the AS as follows:

$$LAD(t,\ \Delta,\ u)(x) = u_0 + \Delta H(t,\ x,\ A) + \langle A,\ x - \bar{x}\rangle \tag{5.2}$$

6. NUMERICAL EXPERIMENTS

As an example we will consider an almost antagonistic game of two coalitions, which can be taken as a model of market competition. Let x, where $0 \leqslant x \leqslant 1$, be the relative part of the capital investment of the first coalition in the first market. Let $(1 - x)$ be the relative part of the capital investment of the first coalition in the second market. Let y, where $0 \leqslant y \leqslant 1$, be the relative capital investment of the second coalition in the first market, and let $(1 - y)$ be the relative capital investment of the second coalition in the second market. We assume that the interests of the coalitions are represented by the payoff matrices

$$A = \begin{Vmatrix} 11 & 2 \\ 3 & 6 \end{Vmatrix},\ B = \begin{Vmatrix} 2 & 4 \\ 5 & 1 \end{Vmatrix}$$

The average payoffs of the coalitions are given by

$$g_1 = C_a xy - \alpha_1 x - \alpha_2 y + a_{22},\ g_2 = C_b xy - \beta_1 x - \beta_2 y + b_{22}$$
$$C_a = 12,\ \alpha_1 = 4,\ \alpha_2 = 3,\ C_b = -6,\ \beta_1 = -3,\ \beta_2 = -4$$

The system describing the dynamics of capital investment can be modelled by the equations

$$\begin{aligned} \dot{x} &= px(1-x)(C_a y - \alpha_1) + (1-p)(-x+u),\ 0 \le u \le 1,\ 0 \le p \le 1 \\ \dot{y} &= qy(1-y)(C_b x - \beta_2) + (1-q)(-y+v),\ 0 \le v \le 1,\ 0 \le q \le 1 \end{aligned} \tag{6.1}$$

The rate of change $\dot{x}$ of the capital investment of the first coalition depends on two factors. We assume that there is a group of firms in the coalition (their relative weight being $p = 0.8$) which apply the criterion of maximizing the current income $g_a(x, y)$. The behaviour of this group is given by the partial derivative $\partial g_a/\partial x = C_a y - \alpha_1$. The rate of capital investment of this group is given by the first term in the first equation in (6.1) [17]. The second term $(1-p)(-x + u)$ describes the rate of investment of those firms that follow the control signal $u(t)$ from the coordinating centre. In the second equation the rate of capital investment of the second coalition is interpreted in a similar way ($q = 0.8$).

Following the approach described earlier in [14, 16], we define the payoff of either coalition by an integral functional with discount coefficient λ. This functional can be regarded as the total payoff in the infinite time interval $[0, +\infty]$

$$J_i = \int_0^{+\infty} \exp(-\lambda t) g_i(x,\ y)dt,\ i = 1,2 \tag{6.2}$$

To construct optimal control interactions in the above game one must consider two identical problems of guaranteed control with respect to J_1 and J_2 [13]. For example, let us consider the first one. The value function $(x, y) \to w_1(x, y)$ is a solution of the Hamilton–Jacobi equation

$$-\lambda w_1(x,\ y) - \frac{\partial w_1}{\partial x}x - \frac{\partial w_1}{\partial y}y + g_a(x,\ y) + \max\left\{0,\ \frac{\partial w_1}{\partial x}\right\} + \min\left\{0,\ \frac{\partial w_1}{\partial y}\right\} = 0 \tag{6.3}$$

Using the FDO (5.1), one can define the AS for solving the Hamilton–Jacobi equation as follows.

Consider the interval $[0, T]$ and the division $\Gamma = [t_0 = 0 < t_1 < \ldots < t_m = T]$ with step Δ. The approximation function W will be defined by the following iteration procedure. We put $W(T, x, y) = 0$. Suppose that $W(t + \Delta, x, y)$ is unknown at some time $t + \Delta$. At a time t the function $W(t, x, y)$ is defined by

$$W(t+\Delta,\ x,\ y) = \Delta g_1(x,\ y) + (1-\lambda\Delta)(u_0 + \langle a_1,\ x-\bar{x}\rangle + \langle a_2,\ y-\bar{y}\rangle +$$
$$+\Delta(a_1(px(1-x)(C_a y-\alpha_1)-(1-p)x) - a_2(qy(1-y)(C_b x-\beta_2)-(1-q)y) +$$
$$+(1-p)\max\{a_1,\ 0\} + (1-q)\min\{a_2,\ 0\}))$$

where (a_1, a_2) is the gradient of the local approximation of $W(t + \Delta, x, y)$ in the neighbourhood of the pseudocentre $(\bar{x}, \bar{y})$. At $t = 0$ we obtain the approximation $W(0, x, y)$ for the solution $w_1(x, y)$.

In Fig. 1 we present a graph of the approximation function $(x, y) \to W(0, x, y)$. Along with computing the value function we constructed the maximizing strategy u^0 of the first coalition. The structure of this strategy is shown in Fig. 2. The square of the phase state splits into two parts. In one of them $u^0 = 1$, the capital is invested in the first market, while in the other part $v^0 = 2$, the capital is invested in the second market.

In a similar way we constructed an approximation of the value function $w_2(x, y)$ and the maximizing strategy v^0 of the second coalition in the game with payoff matrix B.

In Fig. 3 we present the switching lines S_u for the strategy u and S_v for v^0 and show the trajectory TR generated by u and v^0. This trajectory constitutes a basis for dynamic equilibrium in Nash's sense as in [13]. It can be verified that this trajectory converges towards the point of dynamic equilibrium $DE = (0.76, 0.42)$, which can be determined from the system of equations obtained by equating the right-hand sides in (6.1) to zero.

Note that the trajectories of the classical models with replicatory dynamic [17] tend to the static equilibrium point $NE = (0.67; 0.33)$ in Nash's sense or they circulate in the vicinity of this point. The value of the payoff function $g_i(x, y)$, $i = 1, 2$) at DE is much better (much larger) than at the point of statistical equilibrium NE. It follows that the payoff functionals J_i $(i = 1, 2)$ along the trajectories converging to the point of dynamic equilibrium have better values than along the trajectories converging to the point of static equilibrium.

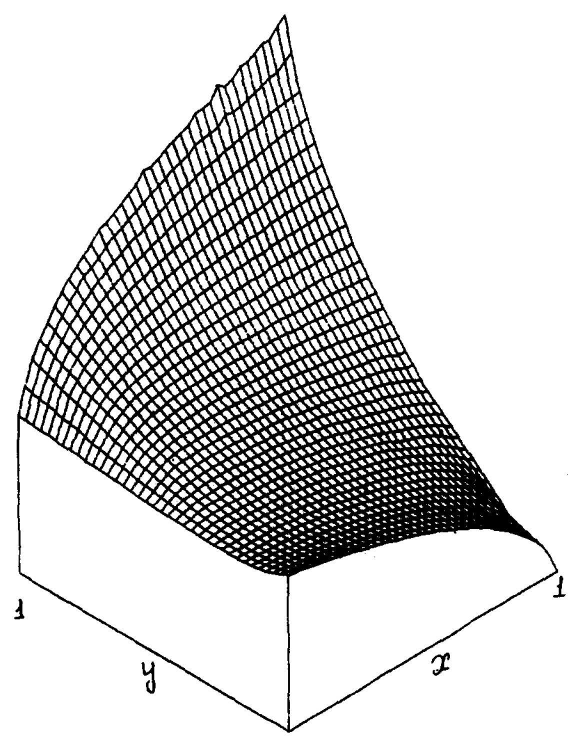

Fig. 1.

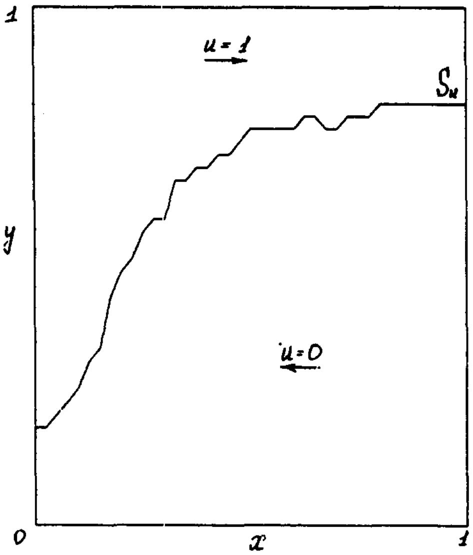

Fig. 2.

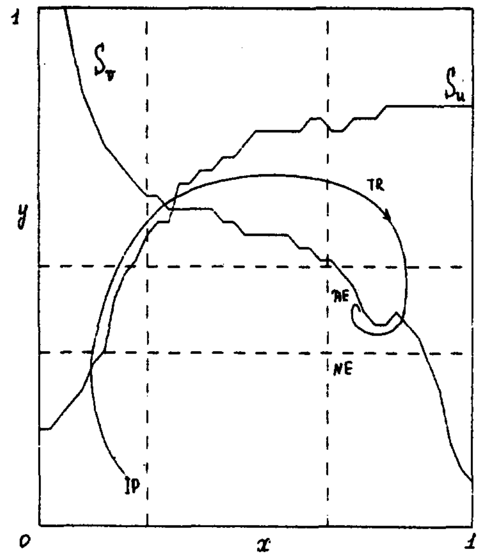

Fig. 3.

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